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In collaboration with IVolatility.com

Variance swaps

Introduction

The goal of this paper is to make a reader more familiar with pricing and hedging variance swaps and to propose some practical recommendations for quoting variance swaps (see section **Conclusions**).

We give basic ideas of variance swap pricing and hedging (for detailed discussion see **[1]**) and apply this analyze to real market data. In the last section we discuss the connection with volatility swaps.

A variance swap is a forward contract on annualized variance, the square of the realized volatility.

$$V_R = \sigma_R^2 = 252 \frac{1}{n-2} \sum_{i=1}^{n-1} \left(\mathrm{Log}\left(\frac{S[t_{i+1}]}{S[t_i]} \right) \right)^2$$

Its payoff at expiration is equal to

$$\mathrm{PayOff} = \sigma_R^2 - K_R^2$$

Where

$S[t_i]$ - the closing level of stock on t_i valuation date.

n - number of business days from the Trade Date up to and including the Maturity Date.

When entering the swap the strike K_R is typically set at a level so that the counterparties do not have to exchange cash flows (‘fair strike’).

Variance swap and option delta-hedging.

Advantage of trading variance swap rather than buying options is that it is pure play on realized volatility no path dependency is involved. Let us recall how path dependency appears in trading P&L of a delta-hedged option position.

If we used implied volatility σ_i on day i for hedging option then P&L at maturity is sum of daily variance spread weighted by dollar gamma.

$$\mathrm{Final\ P\&L} = \frac{1}{2} \sum_{i=1}^{n-1} (r_i^2 - \sigma_i^2 \Delta t)\, \Gamma_i S_i^2$$

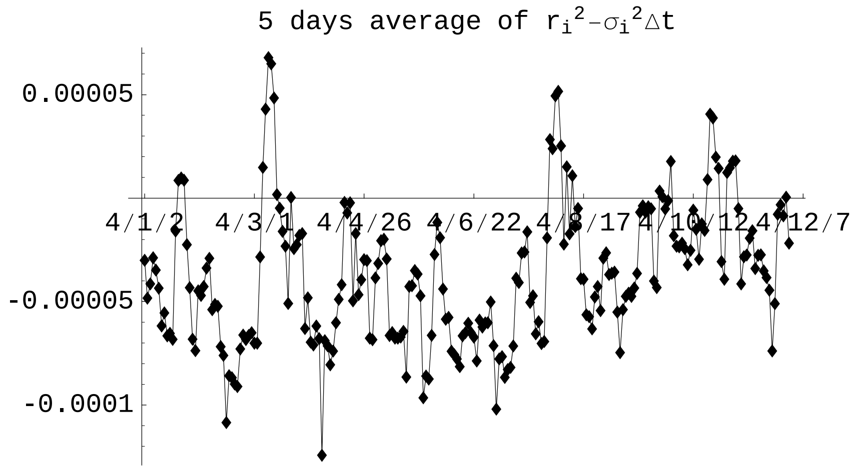
$r_i = \frac{S_{i+1} - S_i}{S_i}$ is stock return on day i

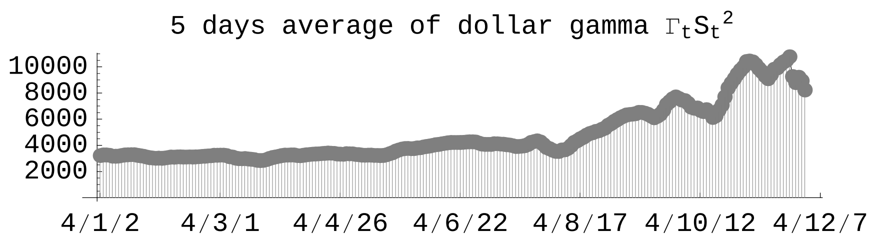
$\Gamma_i S_i^2 = \Gamma[i, S_i]\, S_i^2$ - dollar gamma on day i.

So periods when dollar gamma is high are dominating in P&L.

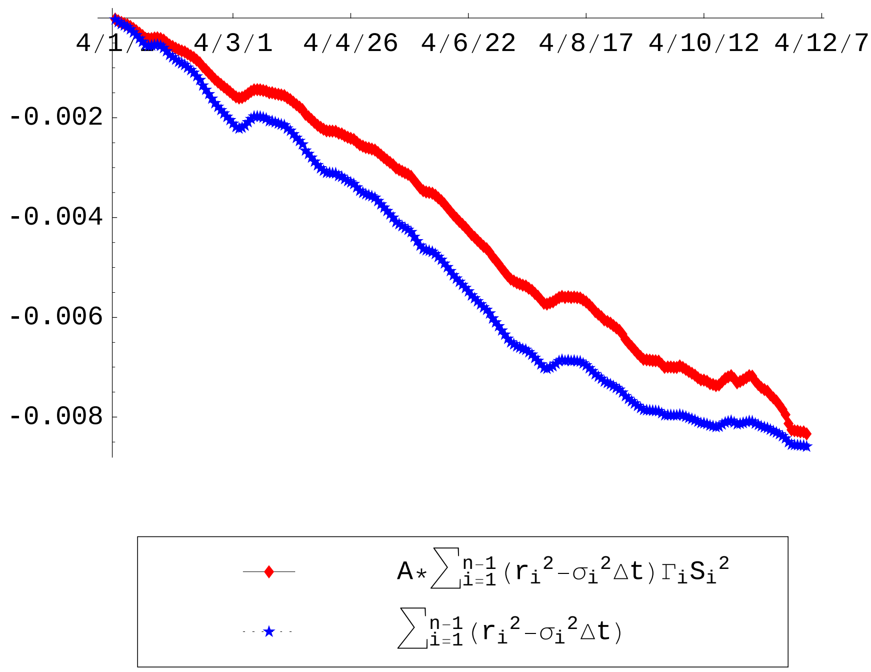
Market example

We use here historical data for the option with one year expiration on SPX500, strike K=1150, observation period from 2004/1/2 to 2004/12/4





5 days average of plain variance spread and variance spread weighted by dollar gamma



Where dollar gamma is normalized by factor $\mathbf{A = 252 / (\sum \Gamma_i S_i^2)}$

Pricing

The fair strike $\mathbf{K_R}$ can be calculated directly from option prices (under assumptions that the underlying follows a continuous diffusion process, see Appendix 1)

$$K_R^2 = \frac{2}{T} \sum_{K_i \le F_T} \frac{\Delta K_i}{K_i^2} e^{r\,T}\,\mathrm{Put}[K_i] + \frac{2}{T} \sum_{K_i > F_T} \frac{\Delta K_i}{K_i^2} e^{r\,T}\,\mathrm{Call}[K_i] \tag{1}$$

where

$F_T = e^{(r-q)\,T}$ is the forward price on the stock at expiration time $\mathbf{T}$,

$\mathbf{r}$ risk-free interest rate to expiration,

$\mathbf{q}$ dividend yield.

Example: VIX

VIX CBOE volatility index it is actually fair strike $\mathbf{K_R}$ for variance swap on S&P500 index.

$$\mathrm{VIX}[T]^2 = \frac{2}{T} \sum_{K_i \le F_T} \frac{\Delta K_i}{K_i^2} e^{r\,T}\,\mathrm{Put}[K_i] + \frac{2}{T} \sum_{K_i > F_T} \frac{\Delta K_i}{K_i^2} e^{r\,T}\,\mathrm{Call}[K_i] - \frac{1}{T}\left(\frac{F_T}{K_0} - 1\right)^2$$

where $\mathbf{K_0}$ is the first strike below the forward index level $\mathbf{F_T}$.

The only difference with formula **(1)** is the correction term

$$-\frac{1}{T}\left(\frac{F_T}{K_0} - 1\right)^2$$

which improves the accuracy of approximation **(1)**.

There two problems in calculation the fair strike of variance swap in from raw market prices:

- We need quotes of European options.

They are available for indexes but not for stocks.

- Number of available option strikes can be not sufficient for accurate calculation.

The first problem can be solved by calculating prices of European options from implied volatilities of listed American options. To overcome the second problem we could use additional strikes and interpolate implied volatilities to this set.

Examples

In this section we calculate prices of variance swap which starts on **n**-th trading day counted from 1.1.2004 with the expiration on 31.12.2004. We compare two values:

The first K_R is calculated based on **40** European options with standardized moneyness **x**

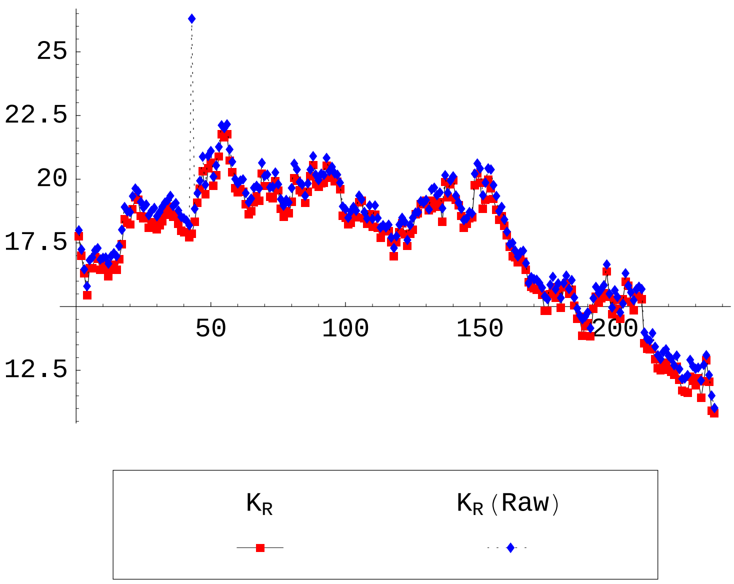
$$x = \mathrm{Log}[K / F_T] \big/ \left(\sigma \sqrt{t}\right)$$

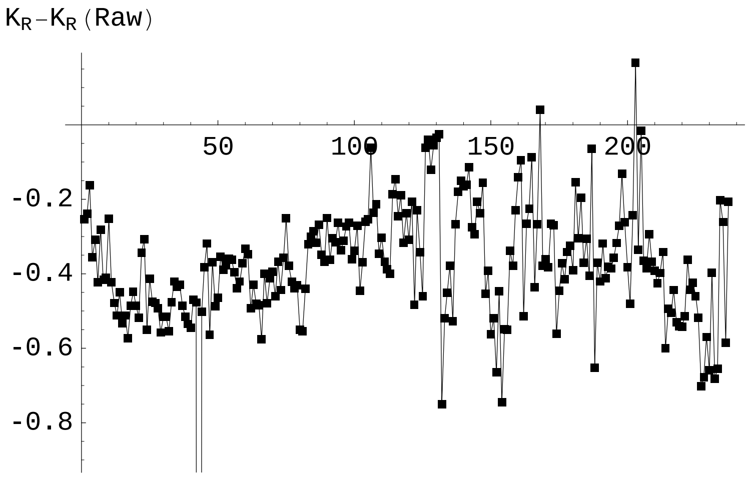
x in range from **-2** to **2** with step **0.1**.

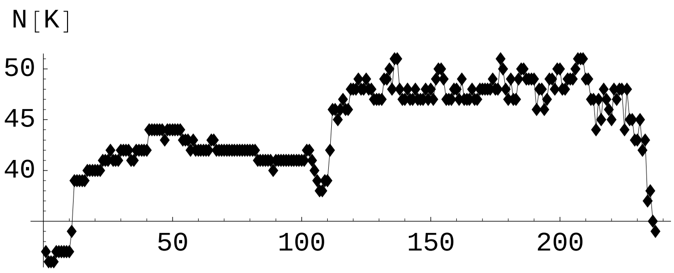
The second value K_R **(Raw)** is calculated based on the raw market prices (either they are European or American). We use only the strikes of options with valid implied volatilities **IV**, i.e. **IV** is in range from **0** to **1**. These strikes we call valid strikes. The number of valid strikes we denote by **N[K].**

SPX

The differences here are quite small less than 1 volatility point. It is natural to expect small difference for SPX because in both cases we used European option and, what is more important, the number of valid strikes is rather large more than **40**.

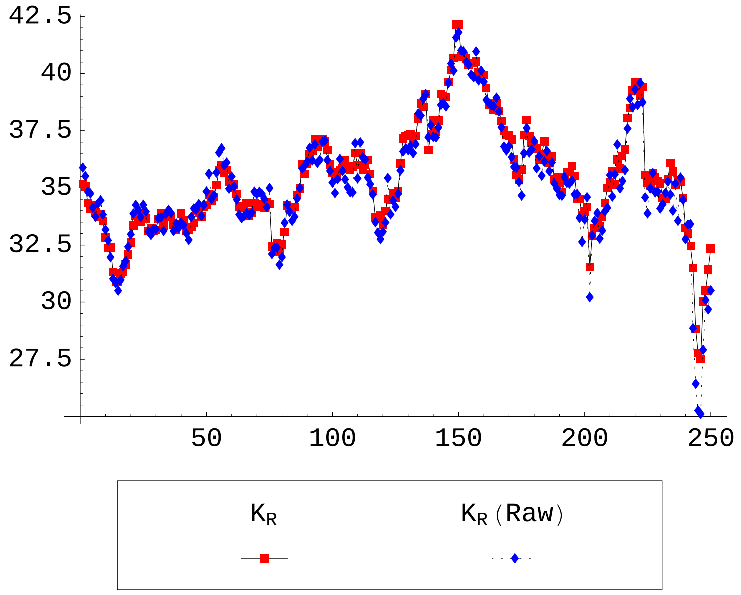


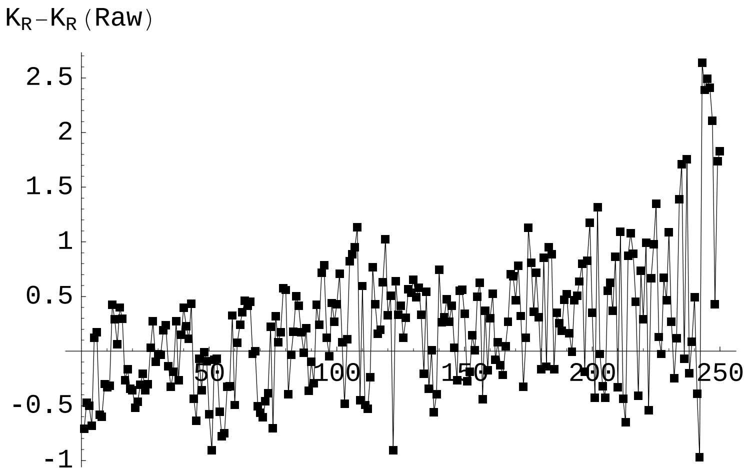


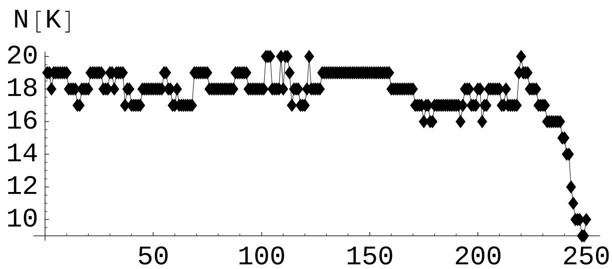


EBAY INC

This example is different. Only American options are available. But still deference are not too large. And we see that divergence happens due to the small number of strikes available for calculation of K_R **(Raw)**. Still the number of valid strikes is relatively large close to **20**.

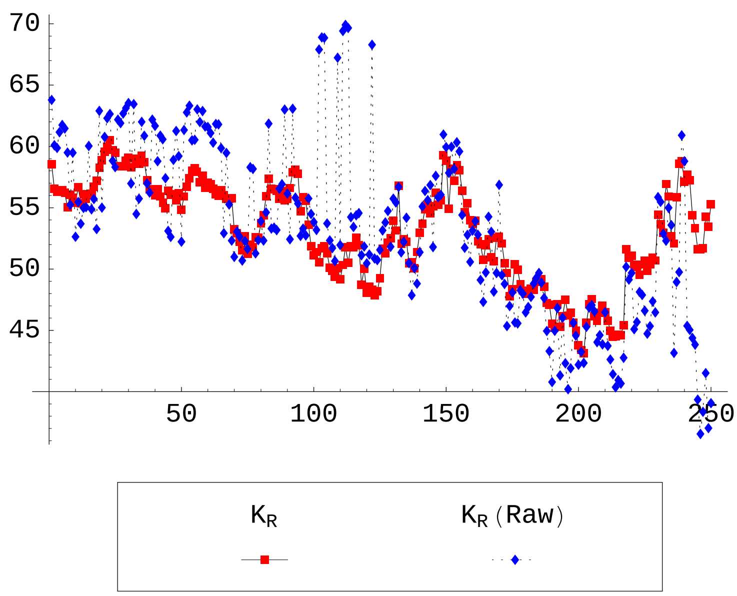


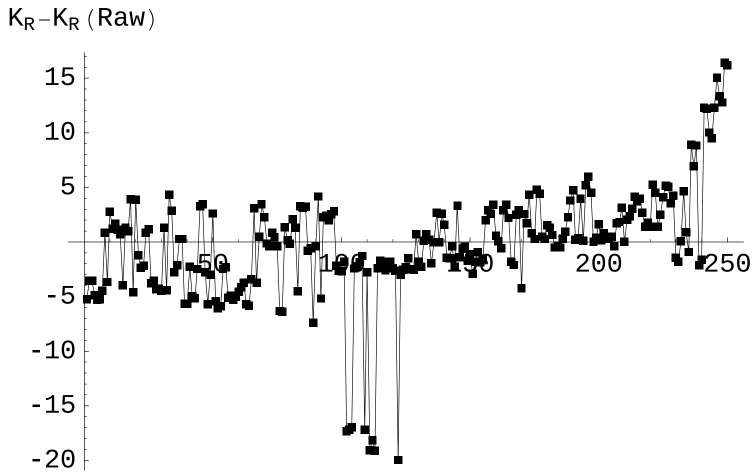


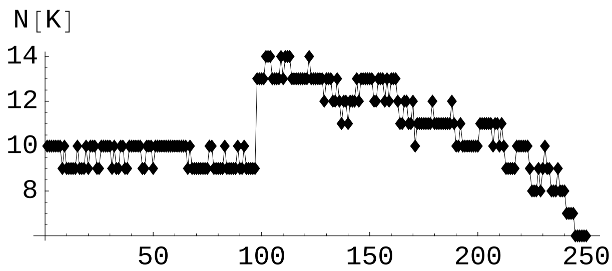


In the following two examples we will see that the quality of approximation **(1)** with raw market prices drops dramatically due to small number of valid strikes.

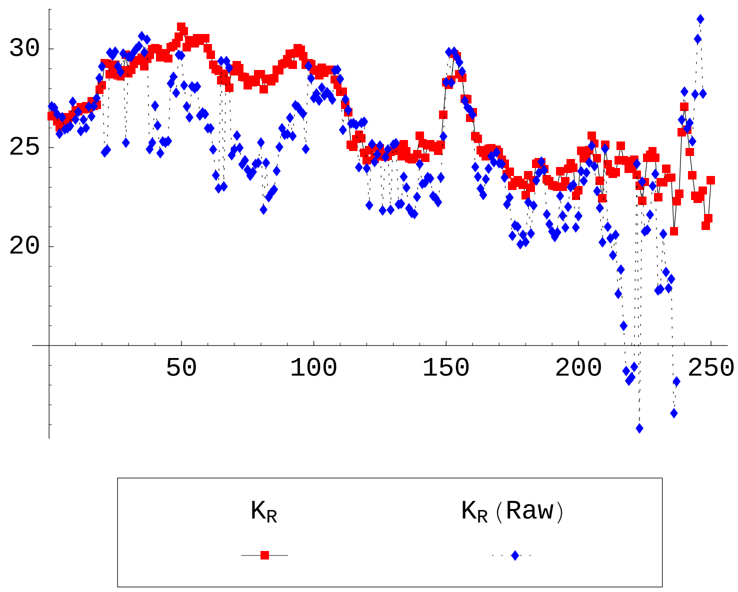
ADVANCED MICRO DEVICES

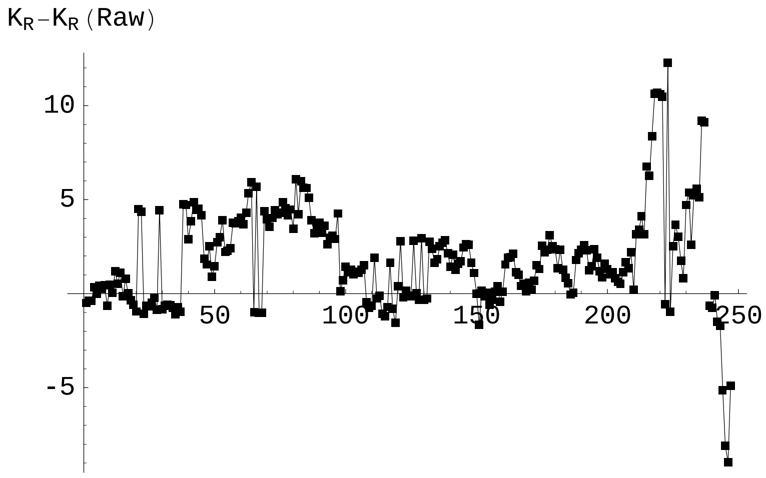


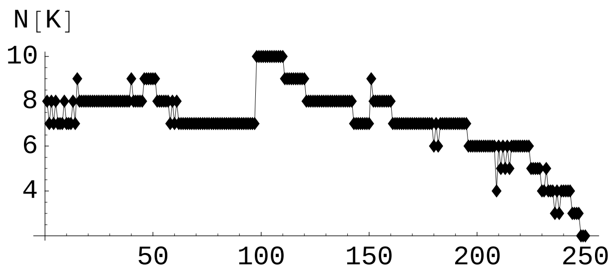




Time Warner INC







Hedging

Replication strategy for variance $V[T]$ follows from the relation (for details see Appendix)

$$\frac{1}{T} V[T] \approx \frac{1}{T} \sum_{i=1}^{n-1} \left(\mathrm{Log}\left(\frac{S[t_{i+1}]}{S[t_i]} \right) \right)^2 \approx \frac{2}{T} \left(\sum_{i=1}^{n-1} \frac{S[t_{i+1}] - S[t_i]}{S[t_i]} - \mathrm{Log}[S_T / S_0] \right)$$

The first term in the brackets

$$\sum_{i=1}^{n-1} \frac{S[t_{i+1}] - S[t_i]}{S[t_i]}$$

can be thought as P&L of continuous rebalancing a stock position so that it is always long $1 / S_t$ shares of the stock.

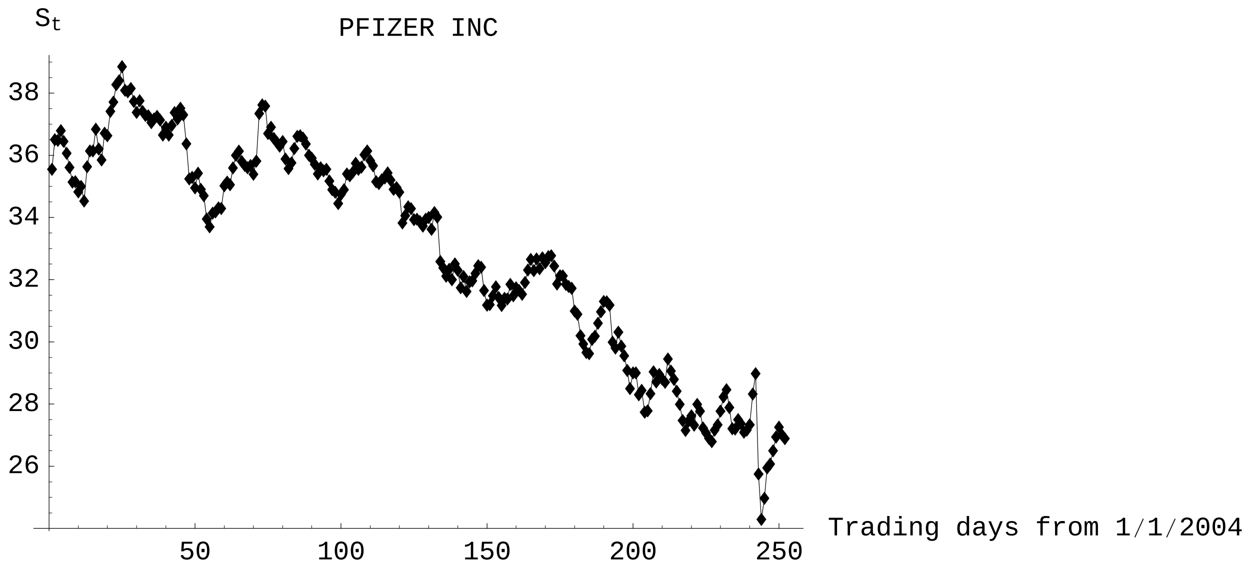
The second term

$$-\mathrm{Log}[S_T / S_0]$$

represent static short position in a contract which pays the logarithm of the total return.

Example

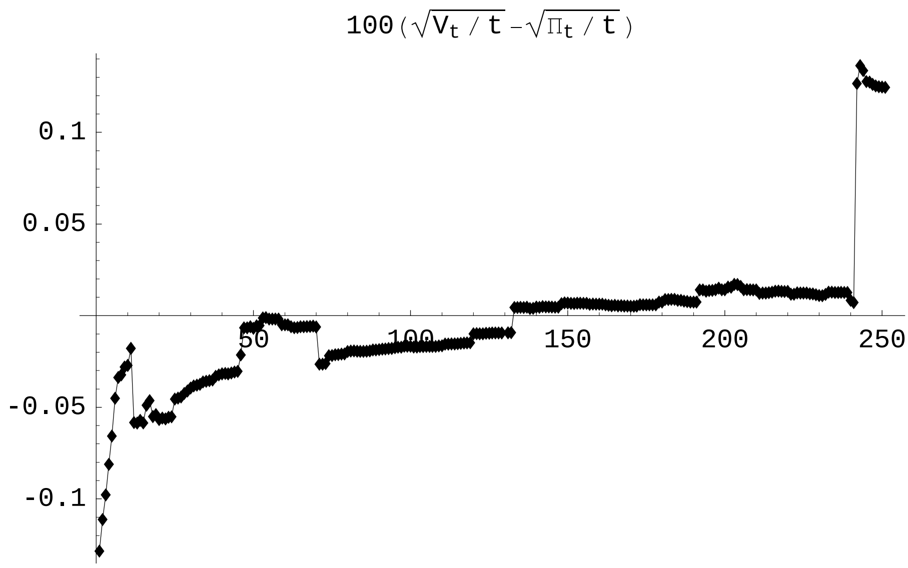
As an example we take the stock prices of PFIZER INC, for one year period from 1/1/2004 to 1/1/2005.



For this stock prices we calculate realized variance V_t and its replication Π_t for each trading day

$$\Pi_t = 2 \sum_{t_i<t} (S[t_{i+1}] - S[t_i]) / S[t_i] - 2\,\mathrm{Log}[S_t / S_0]$$

Difference of realized variance and replication in volatility points



The typical differences in this plot are less than **0.1** volatility point. Large differences in the first **10** days are explained by large ratio of expiration period to time step - **1** day. Large difference around **240**-th trading day appears due to the jump of the stock price.

We see that combination of dynamic trading on stock and log contract can efficiently replicate variance swap.

The payoff of log contract can be replicated by linear combination of puts and calls payoffs (see Appendix)

$$-\mathrm{Log}[S_T / S_0] \approx -\frac{S_T - S_0}{S_0} + \sum_{K_i \le S_0} \frac{\Delta K_i}{K_i^2} (K_i - S_T)_+ + \sum_{K_i > S_0} \frac{\Delta K_i}{K_i^2} (S_T - K_i)_+$$

Hence log contract can be replicated by standard market instruments:

- short position in $1/S$ forward contracts strike at S_0
- long position in $\Delta K_i / K_i^2$ put options strike at K, for all strikes K_i from 0 to S_0,
- long position in $\Delta K_i / K_i^2$ call options strike at K, for all strikes $K_i > S_0$

So this portfolio (with doubled positions) and dynamic trading on stock replicates variance swap. The replication also gives the fair strike value of volatility swap at time t

$$K_R[t]^2 \approx \frac{2}{T} \left(\sum_{t_{i+1} < t} \frac{S[t_{i+1}] - S[t_i]}{S[t_i]} + \mathrm{Log}[F_T[t] / S_t] \right) \tag{2}$$
$$(\text{dynamic trading})$$
$$+ \frac{2}{T} \left(\frac{S_0 - F_T[t]}{S_0} + \sum_{K_i \le S_0} \frac{\Delta K_i}{K_i^2} e^{r(T-t)} \mathrm{Put}[S_t, K_i, T-t] + \sum_{K_i > S_0} \frac{\Delta K_i}{K_i^2} e^{r(T-t)} \mathrm{Call}[S_t, K_i, T-t] \right) \ (\text{static replication})$$

where r is interest rate.

Market examples

In this section we test replication performance on a set of market data.

For each day we calculate price of volatility swap $\sqrt{V_t[T] / T}$ started on **1.1.2004** with expiration on **1.1.2005** and the price of replication strategy $K_R[t]$.

At time t the volatility swap consists of the realized variance and the expected future variance

$$\frac{1}{T} V_t[T] = \underbrace{\frac{1}{T} \sum_{t_{i+1} < t} \left(\mathrm{Log}\left(\frac{S[t_{i+1}]}{S[t_i]} \right) \right)^2}_{\text{realized variance}} +$$
$$\underbrace{\frac{2}{T} e^{r(T-t)} \left(\sum_{K_i \le F_T} \frac{\Delta K_i}{K_i^2} \mathrm{Put}[S_t, K_i, T-t] + \sum_{K_i > F_T} \frac{\Delta K_i}{K_i^2} \mathrm{Call}[S_t, K_i, T-t] \right)}_{\text{expected variance}}$$

For replication we use **40** European options with strikes **K** such that the standardized moneyness **x**

$$x = \mathrm{Log}[K / F_T[0]] \Big/ \left(\sigma \sqrt{T}\right)$$

x is in n range from **-2** to **2** with step **0.1**.

The implied volatilities for these strikes we get by linear interpolation of the market implied volatilities. Then we calculate option prices and get the fair strike value $K_R[t]$ from **Eq.(2)**.

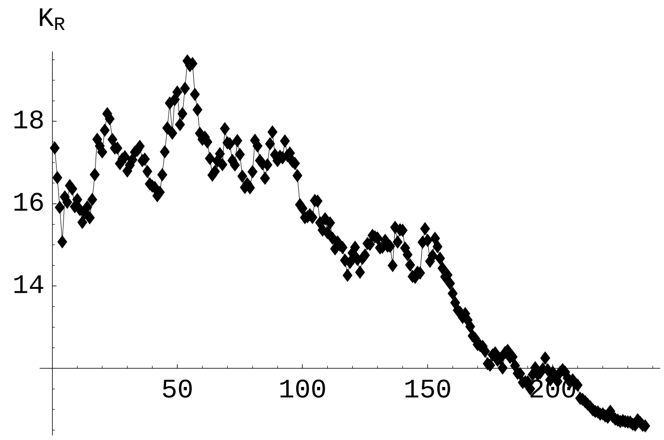
To estimate the performance of replication we define the replication error

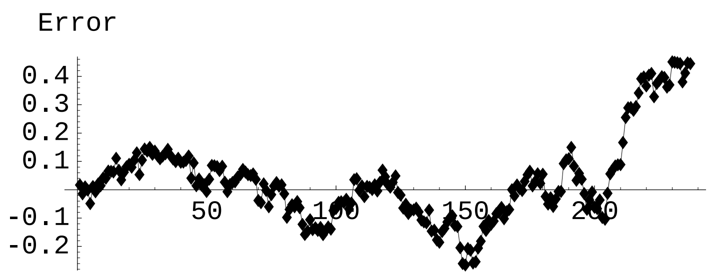
$$\mathrm{Error}[t] = 100 * \left(\sqrt{\frac{1}{T} V_t[T]} - K_R[t]\right)$$

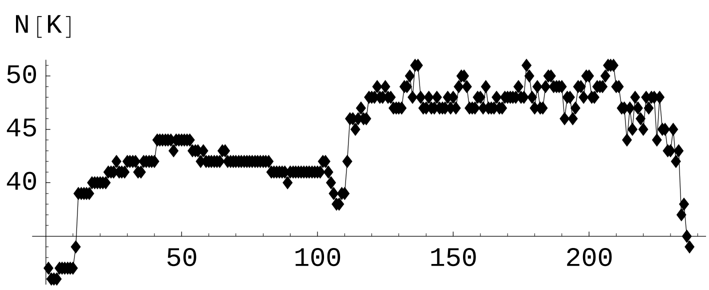
We also plot the number of valid market strikes **N[K]** (strikes with well defined implied volatilities).

Replication Error

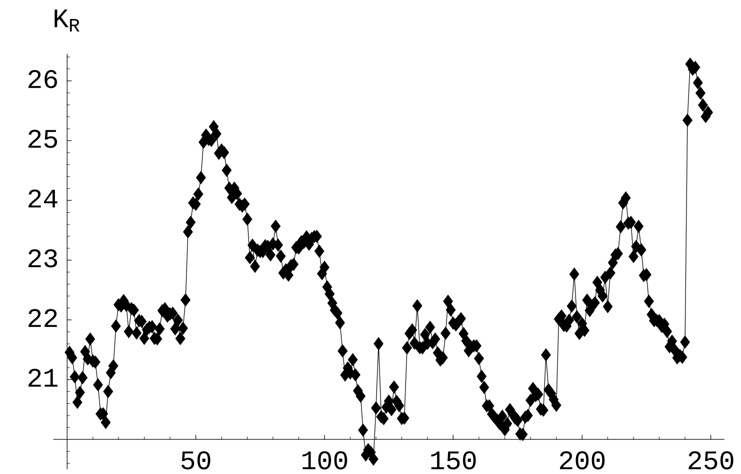
SPX

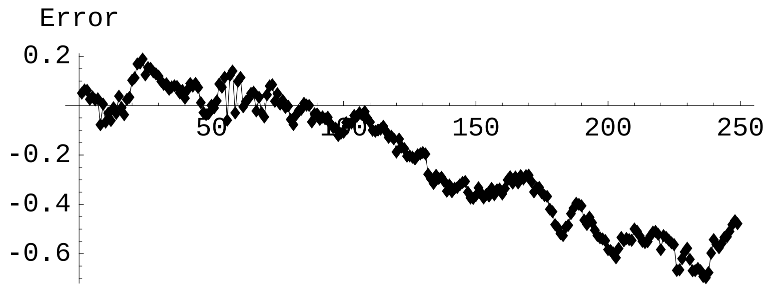


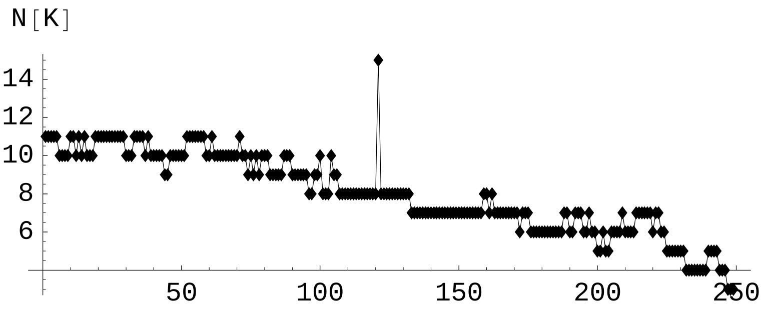




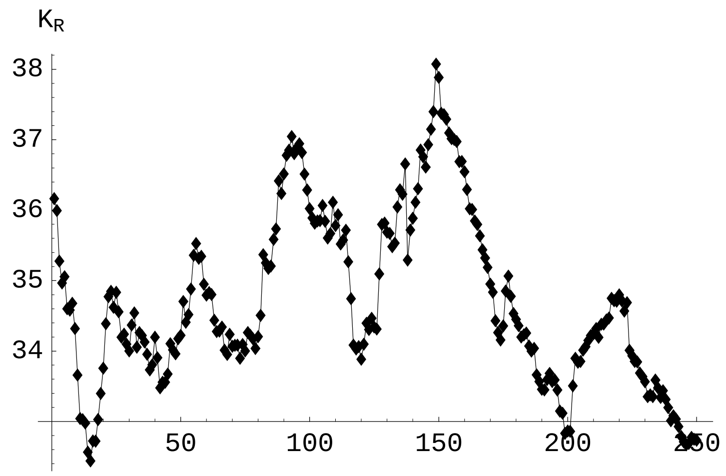
PFIZER INC


K_R
26
25
24
23
22
21
50
100
150
200
250

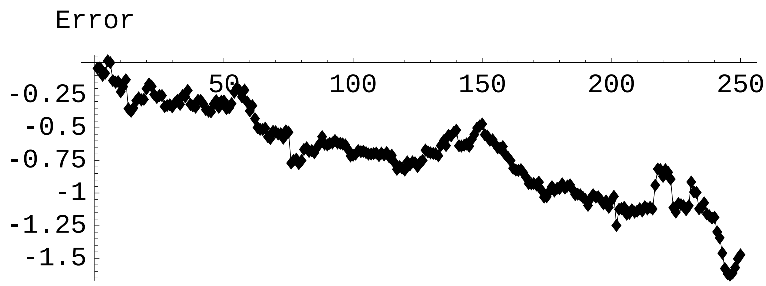

Error
0.2
-0.2
-0.4
-0.6
50
100
150
200
250


N[K]
14
12
10
8
6
50
100
150
200
250

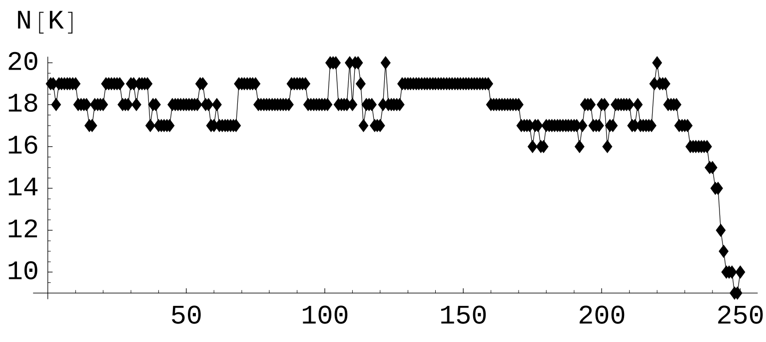
EBAY INC



K_R
38
37
36
35
34
50
100
150
200
250

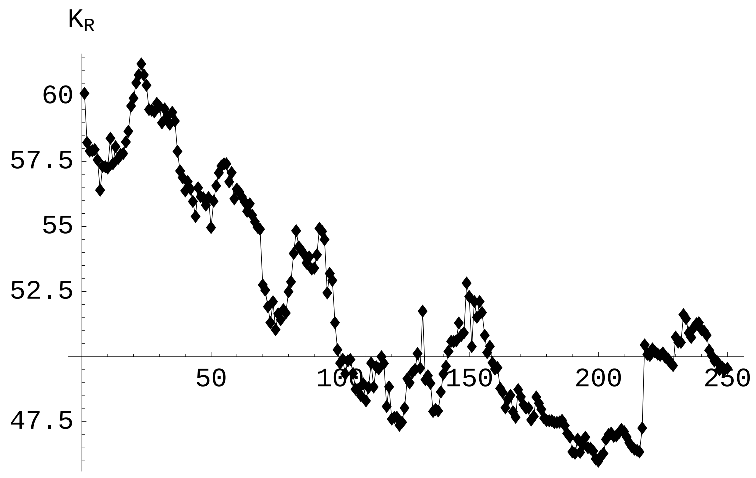


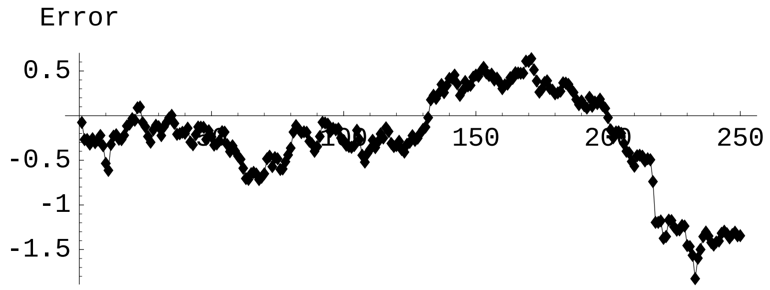
Error
-0.25
-0.5
-0.75
-1
-1.25
-1.5
50
100
150
200
250

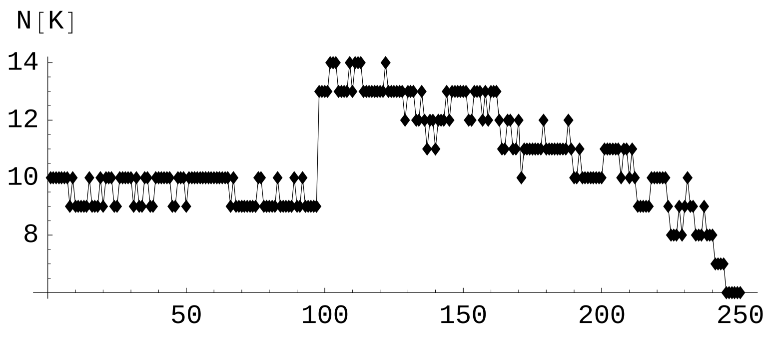


N[K]
20
18
16
14
12
10
50
100
150
200
250

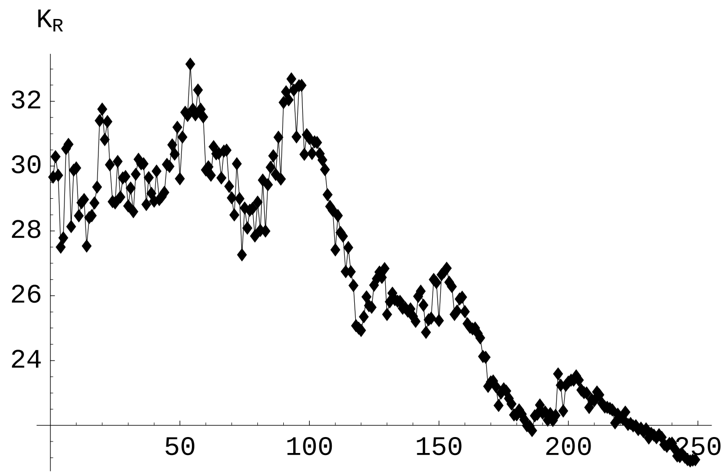
ADVANCED MICRO DEVICES

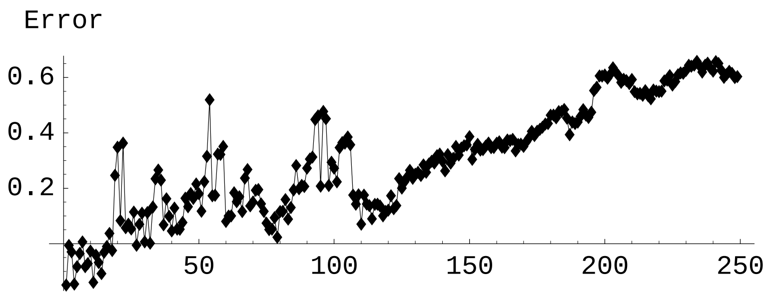


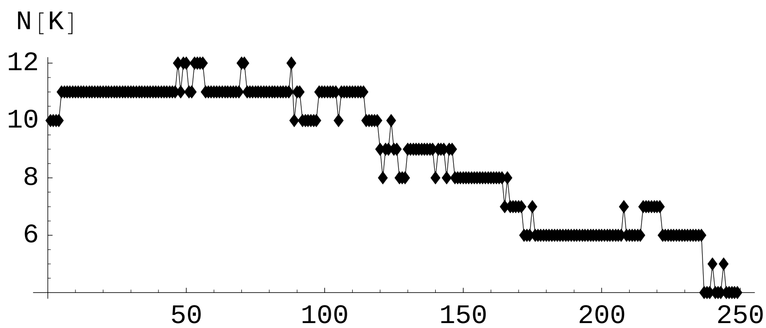




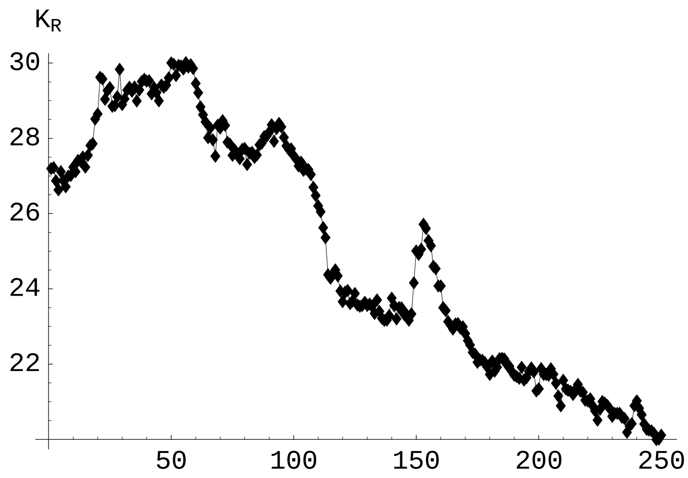
GENERAL MOTORS CORP

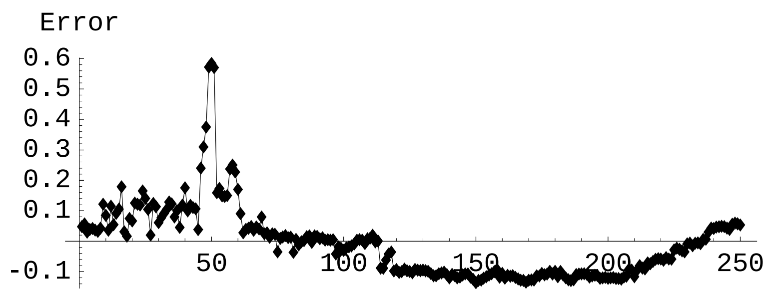


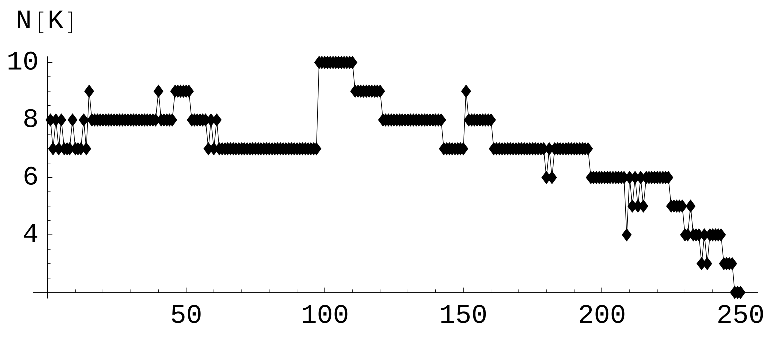




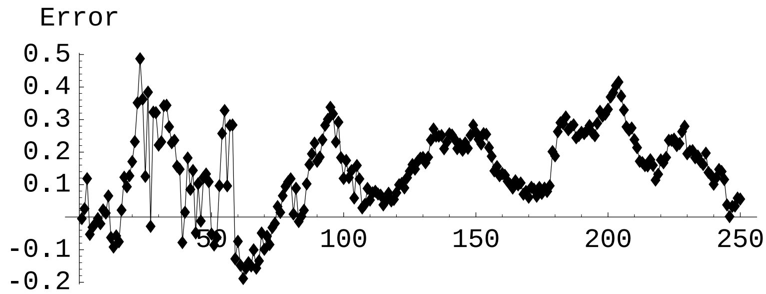
Time Warner Inc.

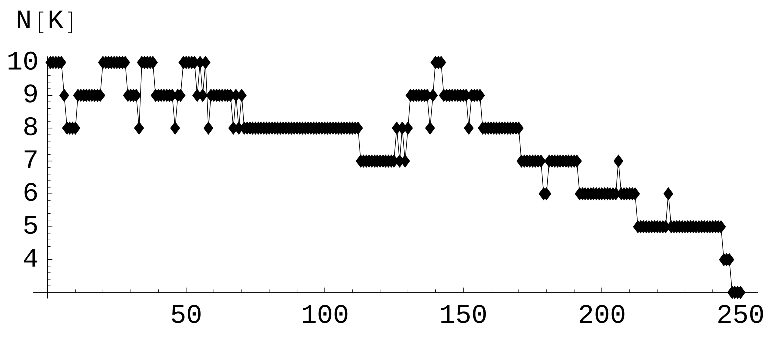




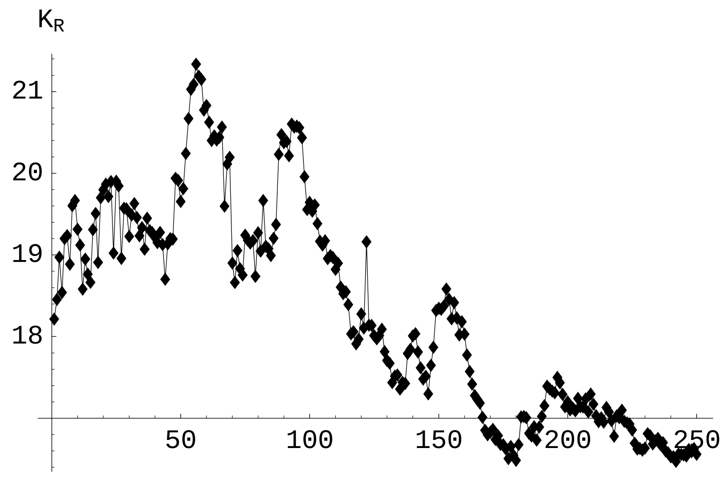


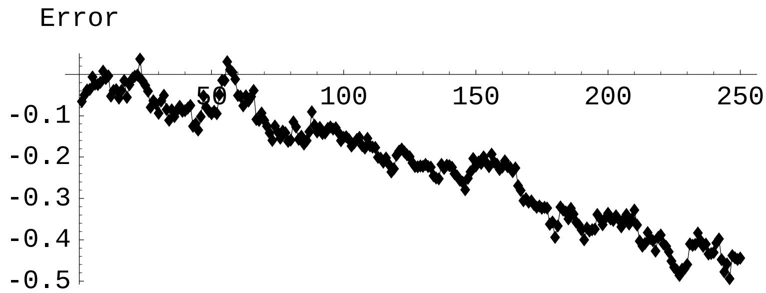
Citigroup Inc

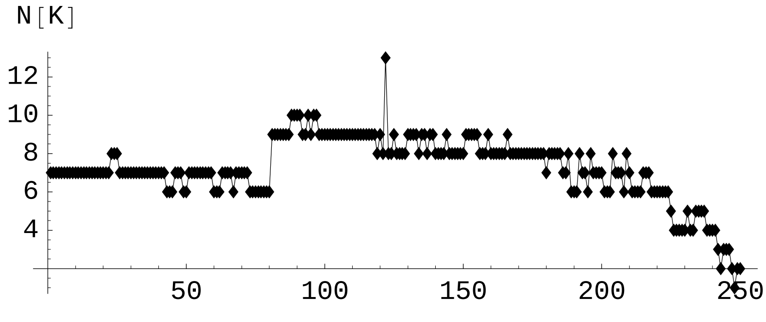




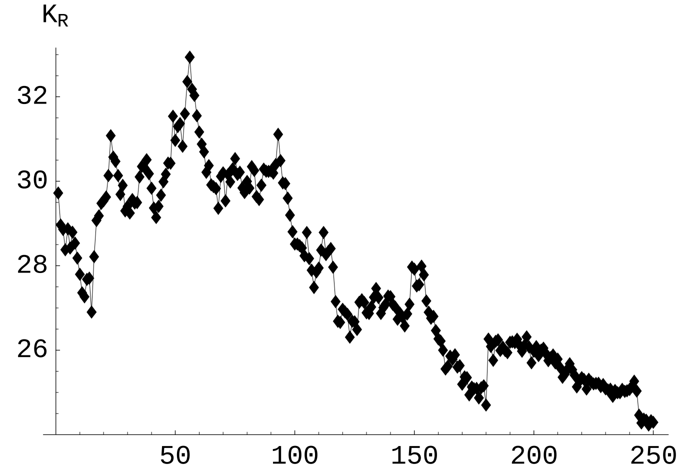
EXXON MOBIL CORP

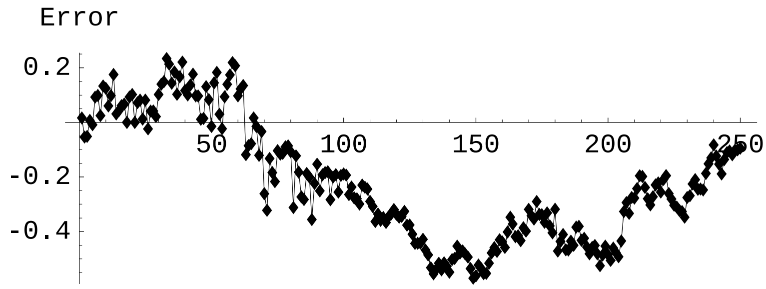


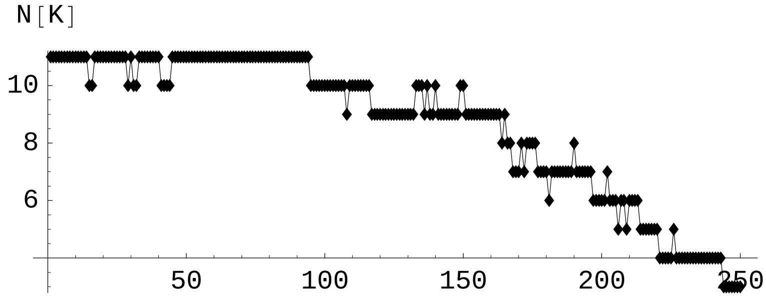




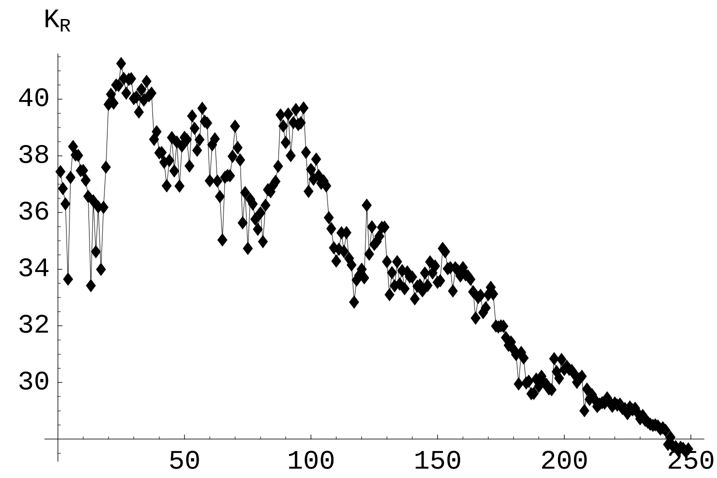
MORGAN STANLEY DEAN WITTER CORP

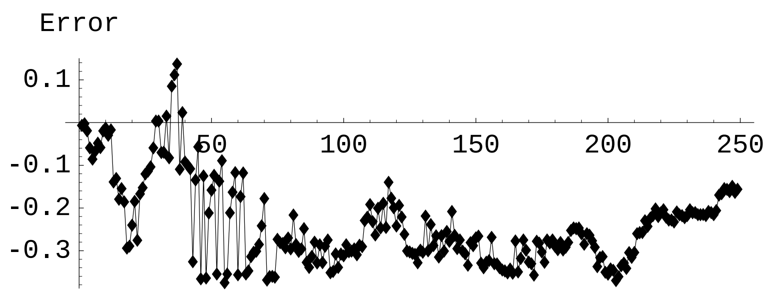


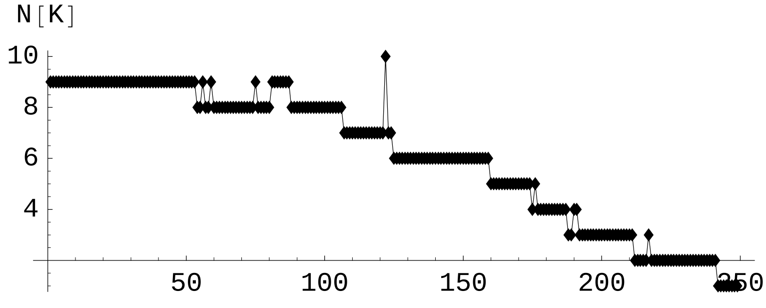




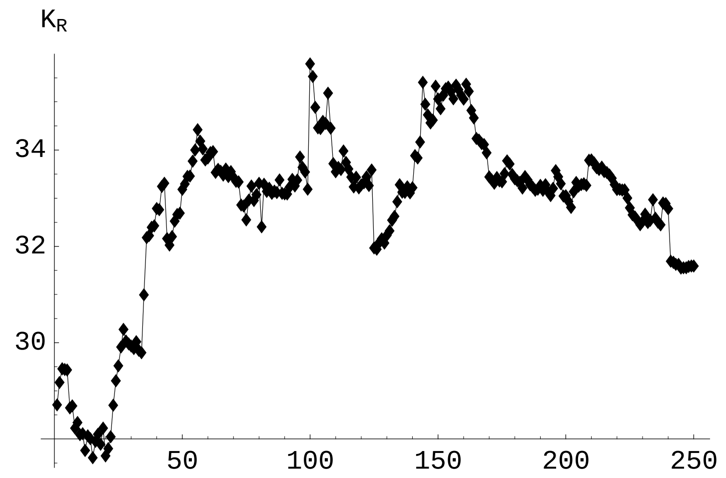
FORD MOTOR CORP

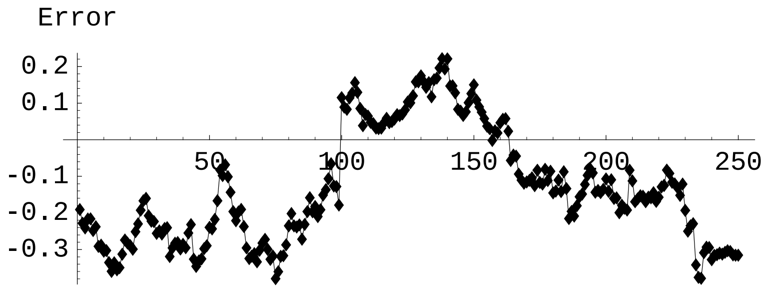


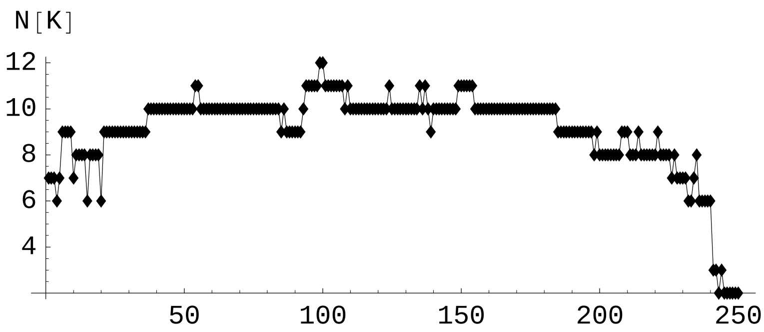




GUIDANT CORP







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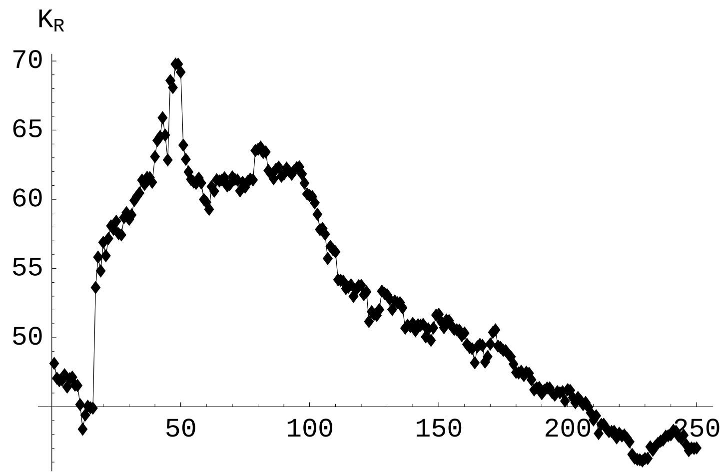

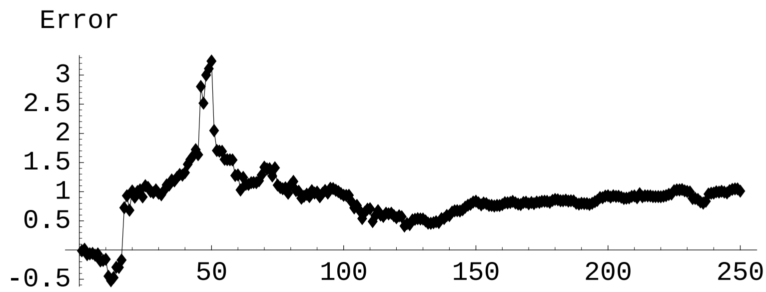

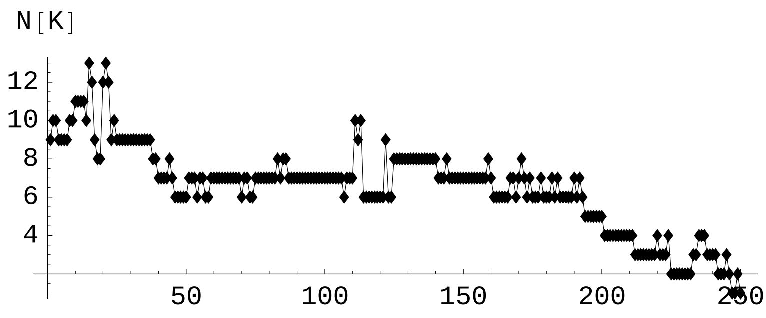

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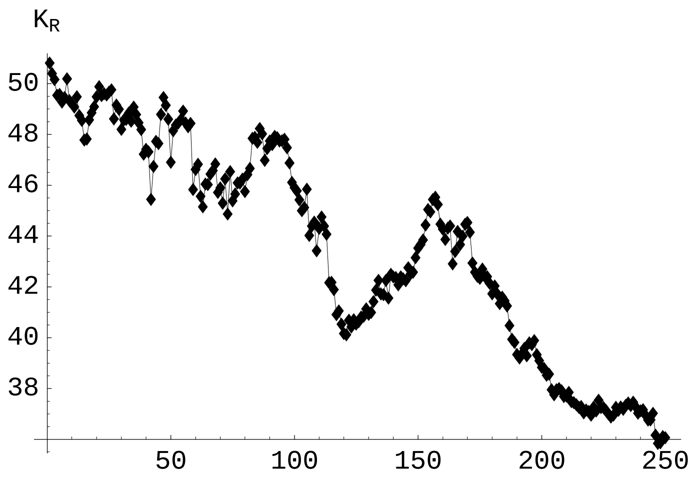
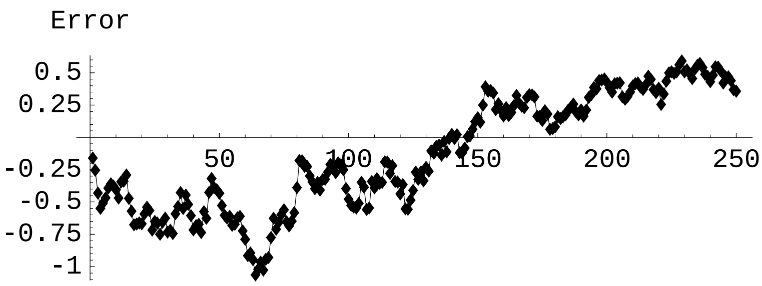
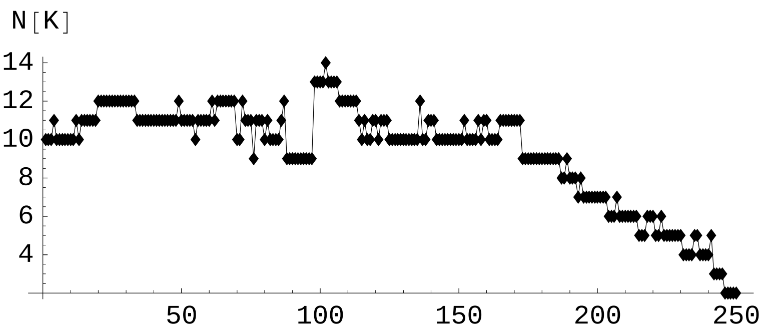






Conclusion

In section **Pricing** we tested two methods for estimation variance swap price. One method is based on the raw option prices. Another is based on a virtual set of European options with implied volatilities extracted from the market data. We saw that the both methods give very close results in case of large number of valid market strikes. But if this number is small (less than 15-10) the second method gives more regular historical prices. Under condition that implied volatilities are available this method is simple, fast and can be recommended for quoting variance swaps.

In section **Hedging** we tested replication strategy for hedging variance swaps. The results confirm that described replication is quite robust, the error of replication is typically less than one volatility point. However practical attractiveness of this method can be restricted by small number of listed options on underlying. In this case trader needs to use OTC market to construct replication portfolio.

Variance and Volatility swaps

It is interesting to compare fair strike level K_R for variance swap with ATM Implied Volatility (IV_{ATM}). This topic is extensively discussed in [2] (see also [3]) with examples of VIX and VXO. This comparison is interesting if we think of IV_{ATM} as an estimate of future volatility σ_R. In this case it is natural to expect that there is a positive spread between K_R and IV_{ATM}. It follows from convexity adjustment argument (see Appendix). Let us recall it.

Suppose the future variance $V[T]$ has mean $\bar{V}$ and variance W (under risk-neutral measure)

$$\bar{V} = E\,V[T],$$
$$W = E\,(V[T] - W).$$

Then

$$K_R = \sqrt{EV[T]} = \sqrt{\bar{V}}$$

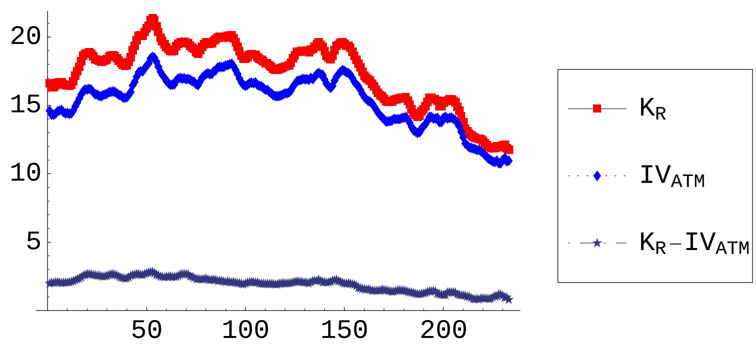
And

$$K_R - \sigma_R \approx \frac{W}{8\,K_R{}^3}$$

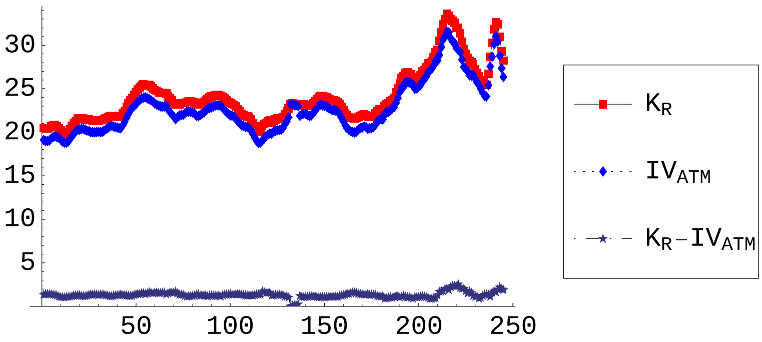
We plot here 5 days moving average of fair strike level K_R calculated for the period (t, T) calculated by Eq. (1) and ATM implied volatility of the same period. Indeed we see quite stable positive spread between K_R and ATM implied volatility. If interpretation of ATM implied volatility as expected future volatility really holds then from this spread we can estimate new parameter W variance of variance. This parameter can be used for price estimates of volatility or variance derivatives.

K_R, σ_R and $K_R - \sigma_R$

SPX



PFIZER INC



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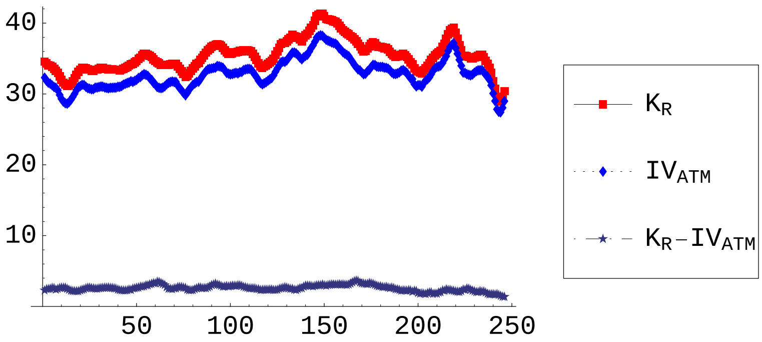
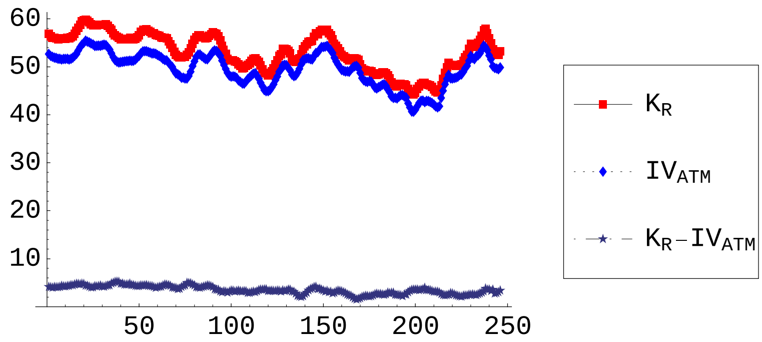
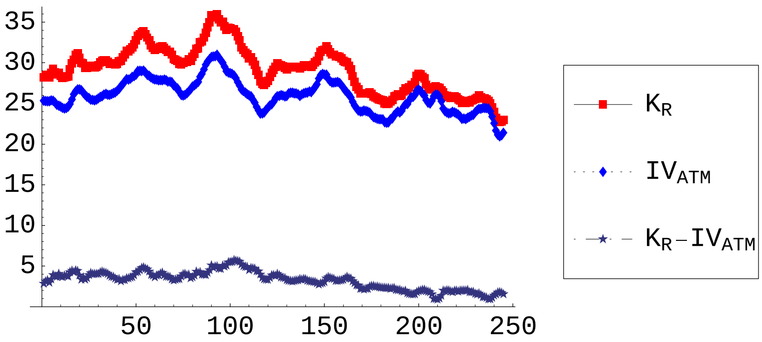


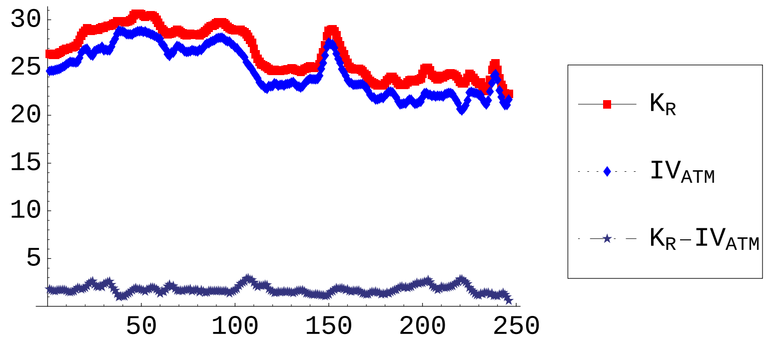
ADVANCED MICRO DEVICES



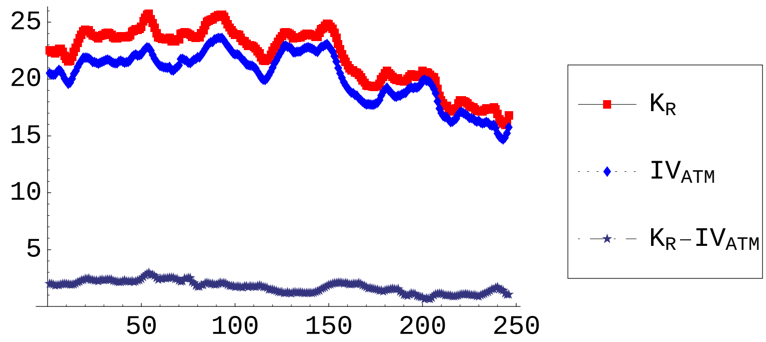
GENERAL MOTORS CORP



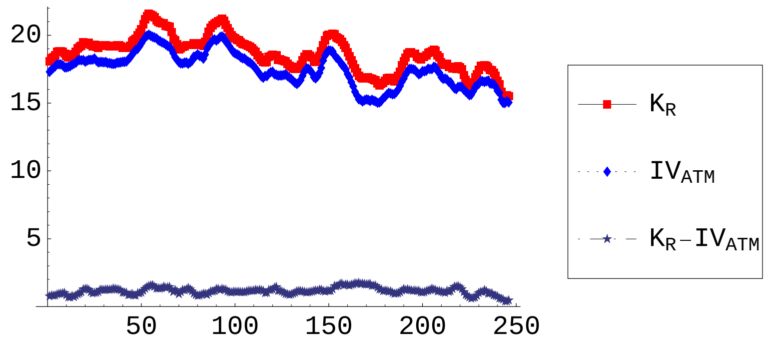
Time Warner INC



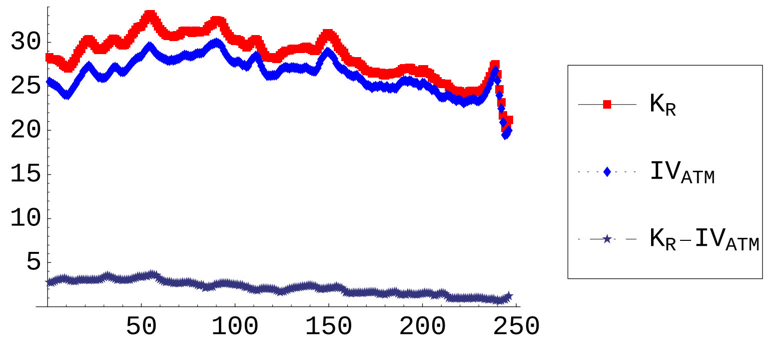
Citigroup INC



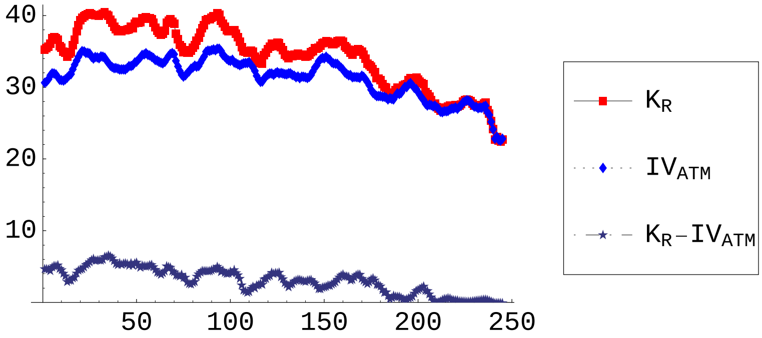
EXXON MOBIL CORP



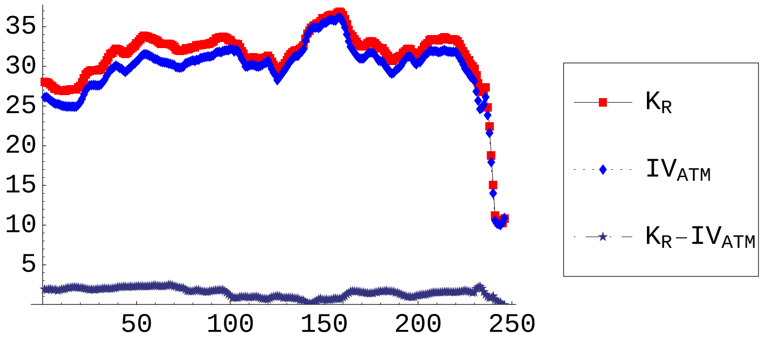
MORGAN STANLEY DEAN WITTER



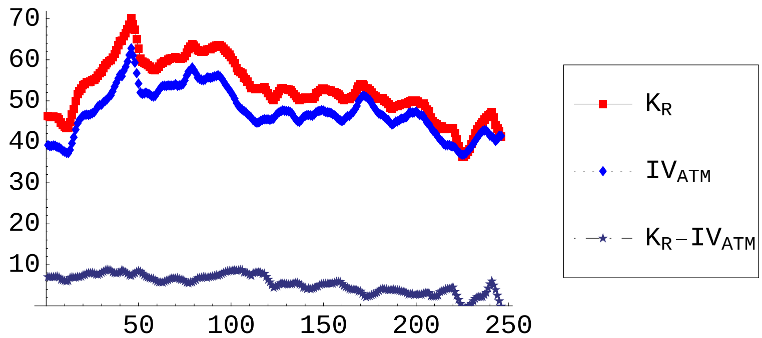
FORD MOTOR



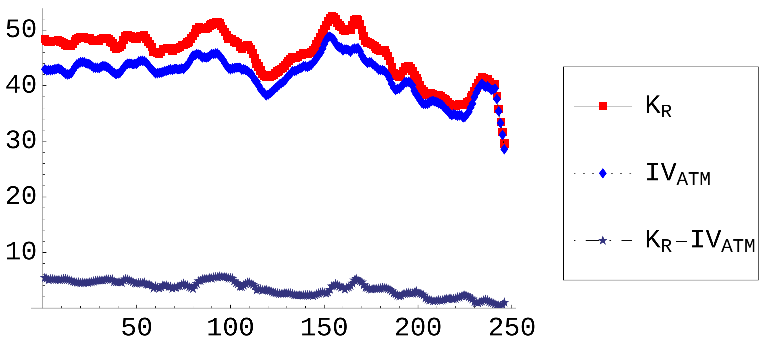
GUIDANT CORP



TENET HEALTHCARE CORP



MICRON TECHNOLOGY INC



References

[1] K. Demeterfi, E. Derman, M. Kamal, J. Zou More Than You Ever Wanted To Know About Volatility Swaps. Quantitative Strategies: Research Notes, Goldman Sachs 1999

[2] P.Carr and L.Wu A Tale of Two Indices *http://www.math.nyu.edu/research/carrp/papers/pdf/vixov_florida3.pdf*

[3] P.Carr and R. Lee Robust replication of volatility derivatives *http://www.math.nyu.edu/research/carrp/papers*

Appendix

Variance Replication

If stock price is follows continuous diffusion with volatility σ_t

$$\frac{dS_t}{S_t} = \mu_t\, dt + \sigma_t\, dW_t$$

Then by Ito's lemma

$$d\,(\mathrm{Log}[S_t]) = \mu_t\, dt + \sigma_t\, dW_t - \frac{1}{2}\,\sigma_t{}^2\, dt$$

Or

$$\frac{1}{2}\,\sigma_t{}^2\, dt = \frac{dS_t}{S_t} - d\,\mathrm{Log}[S_t]$$

Hence for total variance from 0 to T we get

$$V[T] = \int_0^T \sigma_t{}^2\, dt = 2\int_0^T \frac{dS_t}{S_t} - 2\,\mathrm{Log}[S_T/S_0] \qquad (A1.1)$$

Log payoff decomposition

It is easy to check the following identity for any $S>0$

$$\mathrm{Log}[S_T/S] = \frac{1}{S}\,(S_T - S) - \int_0^S \frac{1}{K^2}\,(K - S_T)_+\, dK - \int_S^\infty \frac{1}{K^2}\,(S_T - K)_+\, dK \qquad (A1.2)$$

The fair strike value of variance swap van be calculated by taking expectation of future variance under risk-neutral measure at time t

$$K_R[t]^2 = \frac{1}{T}\,E\,V[T] = \frac{2}{T}\left(E\int_0^T \frac{dS_\tau}{S_\tau} - E\,\mathrm{Log}[S_T/S_0]\right) =$$

$$\frac{2}{T}\left(\int_0^t \frac{dS_\tau}{S_\tau} + (r-q)\,(T-t)\right) - \frac{2}{T}\,E\,\mathrm{Log}[S_T/S_0]$$

Using **(A1.2)** with $S = S_0$

$$E\,\mathrm{Log}[S_T / S_0] =$$
$$\frac{1}{S_0}\,(F_T[t] - S_0) - \int_0^{S_0} \frac{1}{K^2}\, e^{r\,(T-t)}\,\mathrm{Put}[K,\ T - t]\,dK - \int_{S_0}^{\infty} \frac{1}{K^2}\, e^{r\,(T-t)}\,\mathrm{Put}[K,\ T - t]\,dK$$

where

$$F_T[t] = e^{(r-q)\,(T-t)}.$$

Finally we have

$$K_R[t]^2 = \frac{2}{T}\left(\int_0^t \frac{dS_\tau}{S_\tau} + (r - q)\,(T - t)\right) +$$
$$\frac{2}{T}\left(\frac{1}{S_0}\,(F_T[t] - S_0) + \int_0^{S_0} \frac{1}{K^2}\, e^{r\,(T-t)}\,\mathrm{Put}[K,\ T - t]\,dK + \int_{S_0}^{\infty} \frac{1}{K^2}\, e^{r\,(T-t)}\,\mathrm{Put}[K,\ T - t]\,dK\right)$$

Pricing at initial moment, t=0

Expression for the fair strike $K_R = K_R[0]$ at time-0 can be simplified in the following way

$$K_R^2 = \frac{1}{T}\,E\,V[T] = \frac{2}{T}\left(E\int_0^T \frac{dS_t}{S_t} - E\,\mathrm{Log}[S_T / S_0]\right) = \frac{2}{T}\,((r - q)\,T - E\,\mathrm{Log}[S_T / S_0])$$
$$= -\frac{2}{T}\,E\,\mathrm{Log}[S_T / F_T]$$

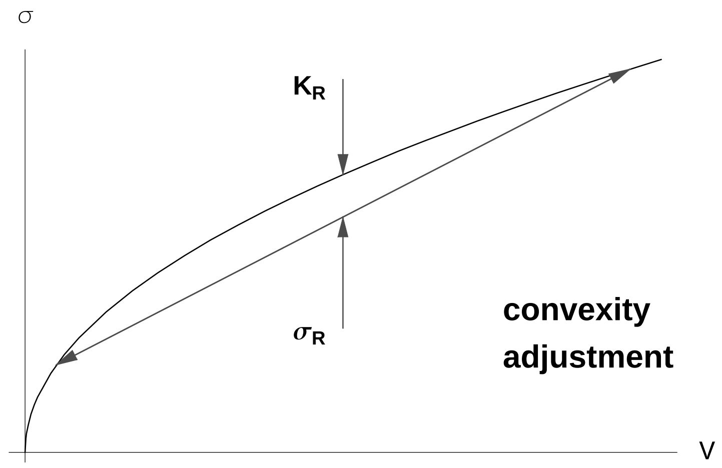
Using **(A1.2)** with $S = F_T$

$$-E\,\mathrm{Log}[S_T / F_T] = \int_0^{F_T} \frac{1}{K^2}\, e^{rT}\,\mathrm{Put}[K]\,dK + \int_{F_T}^{\infty} \frac{1}{K^2}\, e^{rT}\,\mathrm{Call}[K]\,dK$$

Hence

$$K_R^2 = \frac{e^{rT}}{T}\left(\int_0^{F_T} \frac{1}{K^2}\,\mathrm{Put}[K]\,dK + \int_{F_T}^{\infty} \frac{1}{K^2}\, e^{rT}\,\mathrm{Call}[K]\,dK\right)$$

Convexity Adjustment



The expected value of future volatility σ_R is equal to expectation of square root of future variance

$$\sigma[T] = \sqrt{V[T]}$$

$$\sigma_R = E\sigma[T] = E\sqrt{V[T]}$$

Taking second order approximation of square root in $\bar{V}$

$$\sqrt{V[T]} \approx \sqrt{\bar{V}} + \frac{V[T] - \bar{V}}{2\bar{V}^{1/2}} - \frac{(V[T] - \bar{V})^2}{8\bar{V}^{3/2}}$$

And hence

$$\sigma_R = E\sqrt{V[T]} \approx \sqrt{\bar{V}} - \frac{W}{8\bar{V}^{3/2}}$$